

Period functions and cotangent sums

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Abstract

We investigate the period function of $\sum_{n=1}^{\infty} \sigma_a(n) e(nz)$, showing it can be analytically continued to $|\arg z| < \pi$ and studying its Taylor series. We use these results to give a simple proof of the Voronoi formula and to prove an exact formula for the second moments of the Riemann zeta function. Moreover, we introduce a family of cotangent sums, functions defined over the rationals, that generalize the Dedekind sum and share with it the property of satisfying a reciprocity formula. In particular, we find a reciprocity formula for the Vasyunin sum.

1 Introduction

In the well-known theory of period polynomials one constructs a vector space of polynomials associated with a vector space of modular forms. The Hecke operators act on each space and have the same eigenvalues. Thus, either vector space produces the usual degree 2 L-series associated with holomorphic modular forms. In 2001 Lewis and Zagier extended this theory and defined spaces of period functions associated to non-holomorphic modular forms, i.e. to Maass forms and Eisenstein series. Period functions are real analytic functions $\psi(x)$ which satisfy three term relations

$$\psi(x) = \psi(x+1) + (x+1)^{-2s} \psi\left(\frac{x}{1+x}\right) \quad (1)$$

where $s = 1/2 + it$. The period functions for Maass forms are characterized by (1) together with the growth conditions $\psi(x) = o(1/x)$ as $x \rightarrow 0^+$ and $\psi(x) = o(1)$ as $x \rightarrow \infty$; for these $s = 1/2 + ir$ where $1/4 + r^2$ is the eigenvalue of the Laplacian associated with a Maass form. For Eisenstein series, where

the o 's in the above growth conditions are replaced by O 's, Lewis and Zagier consider all $s = 1/2 + it$ but with $t \neq 0$ in their study. They show that ψ , which is initially defined only in the upper half plane, actually has an analytic continuation to all of \mathbb{C} apart from the negative real axis.

In this paper we consider the analogue of these period functions for Eisenstein series but for arbitrary $s \in \mathbb{C}$ and obtain analytic continuations for the period functions to \mathbb{C}' . It turns out that the case $s = 1/2$, i.e. $t = 0$ is especially useful. In this case the arithmetic part of the n -th Fourier coefficient is $d(n)$, the number of divisors of n .

There are several nice applications that are consequences of the analytic continuation of the associated period function, i.e. they are consequences of the surprising fact that the function

$$\sum_{n=1}^{\infty} d(n)e(nz) - \frac{1}{z} \sum_{n=1}^{\infty} d(n)e(-n/z),$$

which apparently only makes sense when the imaginary part of z is positive, actually has an analytic continuation to \mathbb{C}' the slit complex plane (the complex with the negative real axis removed). First, we obtain a new formula for the weighted mean square of the Riemann zeta-function on the critical line:

$$\int_0^{\infty} |\zeta(1/2 + it)|^2 e^{-\delta t} dt.$$

Previously, the best formula for this quantity was a main term plus an asymptotic, but not convergent, series of powers of δ , each term an order of magnitude better than the previous as $\delta \rightarrow 0^+$. Our formula gives an asymptotic series which is also convergent. The situation is somewhat analogous to the situation of the partition function $p(n)$. Hardy and Ramanujan found an asymptotic series for $p(n)$ and subsequently Rademacher gave a series which was both asymptotic and convergent. In both the partition case and our case, the exact formula allows for the computation of the sought quantity to *any* desired degree of precision, whereas an asymptotic series has limits to its precision. Of course, an extra feature of $p(n)$, that is not present in our situation, is that since $p(n)$ is an integer it is known exactly once it is known to a precision of 0.5. However, our formula does have the extra surprising feature that the time required to calculate our desired mean square is basically independent of δ , apart from the intrinsic difficulty of the extra work required just to write down a high precision number δ .

A second application proves a surprising reciprocity formula for the Vasyunin sum, which is a cotangent sum that appears in the Beurling-Nyman criterion for the Riemann Hypothesis. Specifically, the Vasyunin sum appears as part of the exact formula for the twisted mean-square of the Riemann zeta-function on the critical line:

$$\int_0^\infty |\zeta(1/2 + it)|^2 (h/k)^{it} \frac{dt}{\frac{1}{4} + t^2}.$$

The fact that there is a reciprocity formula for the Vasyunin sum is a non-obvious symmetry relating this integral for h/k and the integral for \bar{h}/k where $h\bar{h} \equiv 1 \pmod{k}$. It is not apparent from this integral that there should be such a relationship; our formula reveals a hidden structure.

The reciprocity formula is most simply stated in terms of the function

$$c_0(h/k) = - \sum_{m=1}^{k-1} \frac{m}{k} \cot \frac{\pi m h}{k}$$

defined initially for non-zero rational numbers h/k where h and k are integers with $(h, k) = 1$ and $k > 0$. The reciprocity formula can be simply stated as, “The function

$$c_0\left(\frac{h}{k}\right) + \frac{k}{h} c_0\left(\frac{k}{h}\right) - \frac{1}{\pi h}$$

extends from its initial definition on rationals $x = h/k$ to an (explicit) analytic function on the complex plane with the negative real axis deleted.” This is nearly an example of what Zagier calls a “quantum modular form.”

As a third application, we give a generalization of the classical Voronoi summation formula, which is a formula for $\sum_{n=1}^\infty d(n)f(n)$ where $f(n)$ is a smooth rapidly decaying function. The usual formula proceeds from

$$\sum_{n=1}^\infty d(n)f(n) = \frac{1}{2\pi i} \int_{(2)} \zeta(s)^2 \tilde{f}(s) ds$$

where

$$\tilde{f}(s) = \int_0^\infty f(x)x^{-s} dx.$$

One obtains the formula by moving the path of integration to the left to $\Re s = -1$, say, and then using the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s)$$

of $\zeta(s)$. Here, as usual,

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s).$$

In this way one obtains a leading term

$$\int_0^\infty f(u)(\log u + 2\gamma) du,$$

from the pole of $\zeta(s)$ at $s = 1$, plus another term

$$\sum_{n=1}^\infty d(n)\hat{f}(n)$$

where $\hat{f}(u)$ is a kind of Fourier-Bessel transform of f ; specifically,

$$\hat{f}(u) = \frac{1}{2\pi i} \int_{(-1)} \chi(s)^2 u^{s-1} \tilde{f}(s) ds = \int_0^\infty f(t) C(2\pi\sqrt{tu}) dt$$

with $C(z) = 4K_0(2z) - 2\pi Y_0(2z)$ where K and Y are the usual Bessel functions. By contrast, the period relation implies, for example, that for $0 < \delta < \pi$ and $z = 1 - e^{-i\delta}$

$$\sum_{n=1}^\infty d(n)e(nz) = \frac{1}{4} + 2 \frac{\log(-2\pi iz) - \gamma}{2\pi iz} + \frac{1}{z} \sum_{n=1}^\infty d(n)e\left(\frac{-n}{z}\right) + \sum_{n=1}^\infty c_n e^{-in\delta} \quad (2)$$

where $c_n \ll e^{-2\sqrt{\pi n}}$. This is a useful formula which cannot be readily extracted from the Voronoi formula. In fact, the Voronoi formula is actually an easy consequence of the formula (2). In section 4 we give some other applications of this extended Voronoi formula.

The theory and applications described above are for the period function associated with the Eisenstein series with $s = 1/2$. In this paper we work in a slightly more general setting with $s = a$, an arbitrary complex number. The circle of ideas presented here have other applications and further generalizations, for example to exact formulae for averages of Dirichlet L-functions, which will be explored in future work.

2 Statement of results

For $a \in \mathbb{C}$ and $\Im(z) > 0$, consider

$$\mathcal{S}_a(z) := \sum_{n=1}^{\infty} \sigma_a(n) e(nz),$$

where, as usual, $\sigma_a(n) := \sum_{d|n} d^a$ indicates the sum of the a -th power of the divisors of n and $e(z) := e^{2\pi iz}$. For $a = 2k + 1$, $k \in \mathbb{Z}_{\geq 1}$, $\mathcal{S}_a(z)$ is essentially the Eisenstein series of weight $2k + 2$,

$$E_{a+1}(z) = 1 + \frac{2}{\zeta(-a)} \mathcal{S}_a(z),$$

for which the well known modularity property

$$E_{2k}(z) - \frac{1}{z^{2k}} E_{2k}\left(-\frac{1}{z}\right) = 0$$

holds when $k \geq 2$. For other values of a this equality is no longer true, but we are able to show that the period function

$$\psi_a(z) := E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right) \quad (3)$$

still has some remarkable properties.

Theorem 1. *Let $\Im(z) > 0$ and $a \in \mathbb{C}$. Then $\psi_a(z)$ satisfies the three term relation*

$$\psi_a(z) - \psi_a(z+1) = \frac{1}{(z+1)^{1+a}} \psi_a\left(\frac{z}{z+1}\right) \quad (4)$$

and extends to an analytic function on $\mathbb{C}' := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ via the representation

$$\psi_a(z) = \frac{i}{\pi z} \frac{\zeta(1-a)}{\zeta(-a)} - i \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + i \frac{g_a(z)}{\zeta(-a)},$$

where

$$\begin{aligned} g_a(z) := & -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a) (2\pi z)^{2n-1} + \\ & + \frac{1}{\pi i} \int_{(-\frac{1}{2}-2M)}^{\infty} \zeta(s) \zeta(s-a) \Gamma(s) \frac{\cos \frac{\pi a}{2}}{\sin \frac{\pi(s-a)}{2}} (2\pi z)^{-s} ds, \end{aligned} \quad (5)$$

and M is any integer greater or equal to $-\frac{1}{2} \min(0, \Re(a))$.

Here and throughout the paper equalities are to be interpreted as identities between meromorphic functions in a . In particular, taking the limit $a \rightarrow 0^+$, we have

$$\begin{aligned}\psi_0(z) &= -2 \frac{\log 2\pi z - \gamma}{\pi i z} - 2ig_0(z), \\ g_0(z) &= \frac{1}{\pi i} \int_{(-\frac{1}{2})} \zeta(s)^2 \frac{\Gamma(s)}{\sin \frac{\pi s}{2}} (2\pi z)^{-s} ds = \frac{1}{\pi i} \int_{(-\frac{1}{2})} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s} ds.\end{aligned}$$

We remark that Theorem 1 can be seen as a starting point for the theory of period functions of Maass forms developed by Lewis and Zagier and its proof is based on ideas contained in their paper [12].

For ease of reference, note that (3) can be rewritten in terms of \mathcal{S}_a and g_a as

$$\begin{aligned}\mathcal{S}_a(z) - \frac{1}{z^{a+1}} \mathcal{S}_a\left(-\frac{1}{z}\right) &= \\ &= i \frac{\zeta(1-a)}{2\pi z} - \frac{\zeta(-a)}{2} + \frac{e^{\frac{\pi i(a+1)}{2}} \zeta(a+1) \Gamma(a+1)}{(2\pi z)^{a+1}} + \frac{i}{2} g_a(z).\end{aligned}\tag{6}$$

Another important feature of the function $\psi_a(z)$ comes from the properties of its Taylor series. For example, in the case $a = 0$ one has

$$\frac{\pi i}{2} (1+z) \psi_0(1+z) = -1 - \frac{x}{2} + \sum_{m=2}^{\infty} a_m (-z)^m,$$

with

$$\begin{aligned}a_m &:= \frac{1}{n(n+1)} + 2b_n + 2 \sum_{j=0}^{n-2} \binom{n-1}{j} b_{j+2}, \\ b_n &:= \frac{\zeta(n) B_n}{n}\end{aligned}$$

and where B_{2n} denotes the $2n$ -th Bernoulli number. In particular, the values a_m are rational polynomials in π^2 . The terms involved in the definition of a_m are extremely large, since

$$b_{2n} \sim \frac{B_{2n}}{2n} \sim (-1)^{n+1} 2 \sqrt{\frac{\pi}{n}} \left(\frac{n}{\pi e}\right)^{2n}$$

as $n \rightarrow \infty$, though there is a lot of cancellation; for example, for $m = 20$ one has

$$\begin{aligned} a_m &= \frac{1}{420} + \frac{\pi^2}{36} - \frac{19\pi^4}{600} + \frac{646\pi^6}{19845} - \frac{323\pi^8}{1500} + \frac{4199\pi^{10}}{343035} + \\ &\quad - \frac{154226363\pi^{12}}{36569373750} + \frac{1292\pi^{14}}{1403325} - \frac{248571091\pi^{16}}{2170943775000} + \\ &\quad + \frac{1924313689\pi^{18}}{288905366499750} - \frac{30489001321\pi^{20}}{252669361772953125} \\ &= 0.0499998087\dots \end{aligned}$$

Notice how close this number is to $\frac{1}{20}$; this observation can be made for all m and we can actually prove that

$$a_m - \frac{1}{m} \sim 2^{\frac{5}{4}} \pi^{\frac{3}{4}} \frac{e^{-2\sqrt{\pi m}}}{m^{\frac{3}{4}}} \sin\left(2\sqrt{\pi m} + \frac{3}{8}\pi\right),$$

confirming the asymptotics conjectured by Zagier in a private communication. Zagier conjectured also an asymptotic series for $a_{0,m}$ of the shape

$$a_m = \frac{1}{m} + e^{-2\sqrt{\pi m}} \sum_{\substack{k=3, \\ k \equiv 1 \pmod{2}}}^{L-1} \frac{C_k \sin(2\sqrt{\pi m} + \theta_k)}{m^{\frac{k}{4}}} + O\left(m^{-\frac{L}{4}}\right),$$

an assertion that may be proven by the same techniques used to prove the asymptotics.

Similar results hold for the Taylor series at any point τ in the half plane $\Re(\tau) > 0$ and for any $a \in \mathbb{C}$. We give a proof in the following theorem, using g_a instead of ψ_a to simplify slightly the resulting formulae.

Theorem 2. *Let $\Re(\tau) > 0$ and for $|z| < |\tau|$, let*

$$g_a(\tau + z) := \sum_{m=0}^{\infty} \frac{g_a^{(m)}(\tau)}{m!} z^m$$

be the Taylor series of $g_a(z)$ around τ . Then

$$\begin{aligned} \frac{g_a^{(m)}(1)}{m!} &= \sum_{\substack{2n-1+k=m, \\ n,k \geq 1}} (-1)^{n+m} B_{2n} \zeta(1-2n-a) \frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!} 2(2\pi)^{2n-1} + \\ &+ (-1)^m \cot \frac{\pi a}{2} \zeta(-a) \frac{\Gamma(1+a+m)}{\Gamma(1+a)m!} + \\ &+ (-1)^m \left(\frac{\Gamma(1+a+m)}{\Gamma(a)(m+1)!} - 1 \right) \frac{\zeta(1-a)}{\pi}, \end{aligned} \quad (7)$$

and in particular if $a \in \mathbb{Z}_{\leq 0}$, $(a, m) \neq (0, 0)$, then $\pi g_a^{(m)}(1)$ is a rational polynomial in π^2 . Moreover,

$$\begin{aligned} \frac{g_a^{(m)}(\tau)}{m!} &\sim \cos\left(\frac{\pi a}{2}\right) \frac{2^{\frac{7}{4}-\frac{a}{2}}}{\pi^{\frac{3}{4}+\frac{a}{2}}} \frac{e^{-2\sqrt{\pi\tau m}}}{m^{\frac{1}{4}-\frac{a}{2}} \tau^{m+\frac{3}{4}+\frac{a}{2}}} \times \\ &\times \cos\left(2\sqrt{\pi\tau m} - \frac{\pi}{8}(2a-1) + (\tau+m)\pi\right), \end{aligned} \quad (8)$$

as $m \rightarrow \infty$.

Some of the ideas used in the proofs of Theorem 1 and 2 can be easily generalized to a more general setting. For example, let $G(s)$ be a meromorphic function on $1-\omega \leq \Re(s) \leq \omega$ for some $1 < \omega < 2$ with no poles on the boundary and assume $|F(\sigma + it)| \ll_{\sigma} e^{(\frac{\pi}{2}-\eta)|t|}$ for some $\eta > 0$. Let

$$\begin{aligned} W_+(z) &:= \frac{1}{2\pi i} \int_{(\omega)} F(s) \Gamma(s) (-2\pi i z)^{-s} ds, \\ W_-(z) &:= \frac{1}{2\pi i} \int_{(\omega)} F(1-s) \Gamma(s) (-2\pi i z)^{-s} ds, \end{aligned} \quad (9)$$

for $\frac{\pi}{2} - \eta < \arg z < \frac{\pi}{2} + \eta$. (Notice that these functions are essentially convolutions of the exponential function and the Mellin transform of $F(s)$.) Then we have

$$\sum_{n=1}^{\infty} d(n) W_+(nz) - \frac{1}{z} \sum_{n=1}^{\infty} d(n) W_-\left(-\frac{n}{z}\right) = R(z) + k(z), \quad (10)$$

where

$$R(z) := \operatorname{Res}_{1-\omega < \Re(s) < \omega} (F(s) \Gamma(s) \zeta(s)^2 (-2\pi iz)^{-s}),$$

$$k(z) := \frac{1}{2\pi} \int_{(1-\omega)} F(s) \frac{\zeta(s) \zeta(1-s)}{\sin \pi s} z^{-s} ds$$

and $k(z)$ is holomorphic on $|\arg(z)| < \frac{\pi}{2} + \eta$. Moreover, if we assume that $F(s)$ is holomorphic on $\Re(s) < 1 - \omega$, then it follows that the Taylor series of $k(z)$ converges very fast,

$$\frac{k^{(n)}(\tau)}{n!} \ll n^{-B} |\tau|^{-n}$$

for any $B > 0$ and τ such that $|\arg \tau| < \eta$. Also, $W_-(z)$ decays faster than any power of z at infinity and so the second sum in (10) is rapidly convergent and is very small if we let z goes to zero in $|\arg z| < \eta$. In section 4 we will give an explicit example; a subsequent paper will elaborate on this.

The Voronoi summation formula is an important tool in analytic number theory; in its simplest form, it states that, if $f(u)$ is a smooth function of compact support, then

$$\sum_{n=1}^{\infty} d(n) f(n) = \sum_{n=1}^{\infty} d(n) \hat{f}(n) + \int_0^{\infty} f(t) (\log t + 2\gamma) dt + \frac{f(0)}{4}, \quad (11)$$

where

$$\hat{f}(x) := 4 \int_0^{\infty} f(t) \left(K_0(4\pi\sqrt{tx}) - \frac{\pi}{2} Y_0(4\pi\sqrt{tx}) \right) dt.$$

This formula can be deduced from (10) (or also directly from (6)) as a very easy corollary. Actually, Voronoi's formula can be interpreted as a version of the formula (6) confined to the positive real axis. If we get rid of this limitation and we use directly the period formula (6), we are able to obtain interesting results also for weight functions of the shape $f(u) = e^{-\delta u}$, for which the Voronoi summation formula fails to give a useful formula. (Try it!) Thus, we have a generalization of Voronoi's formula.

The use of a weight function of the shape $e^{-\delta u}$ is fundamental to investigate the smoothly weighted second moment of the Riemann zeta function,

$$L_{2k}(\delta) := \int_0^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} e^{-\delta t} dt,$$

in the case $k = 1$. These integrals play a major role in the theory of the Riemann zeta function and getting good upper bounds on their growth as $\delta \rightarrow 0^+$ would imply the Lindelöf hypothesis. Unfortunately, the only two value of k for which the asymptotics are known are $k = 1$ (Hardy and Littlewood, [7]) and $k = 2$ (Ingham, [8]); for other values we have just conjectures (see [5], [6] and [10]). For $k = 1$, it is easy to see that the smooth moment is strictly related to the sum $\mathcal{S}_0(-e^{-i\delta})$ and, from this, it is easy to deduce an asymptotic expansion for $L_{2k}(\delta)$. This classical asymptotic series is not convergent. Here we replace the series by two series, each of which are absolutely convergent asymptotic series. (See also Motohashi [13]). The following theorem provides a new exact formula for $L_2(\delta)$, by applying Theorem 1 and 2 to $\mathcal{S}_0(-e^{-i\delta})$.

Theorem 3. *For $0 < \Re(\delta) < \pi$, we have*

$$L_2(\delta) = \frac{\gamma - \log 2\pi\delta}{2 \sin \frac{\delta}{2}} + \frac{\pi i}{\sin \frac{\delta}{2}} \mathcal{S}_0\left(\frac{-1}{1 - e^{-i\delta}}\right) + h(\delta) + k(\delta),$$

where $k(\delta)$ is analytic in $|\Re(\delta)| < \pi$ and $h(\delta)$ is C^∞ in \mathbb{R} and holomorphic in

$$\mathbb{C}'' := \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid x \in 2\pi\mathbb{Z}, y \geq 0\}.$$

Moreover, $h(0) = 0$ and, if $\Im(\delta) \leq 0$,

$$h(\delta) = i \sum_{n \geq 0} h_n e^{-i(n + \frac{1}{2})\delta},$$

with

$$h_n = 2^{\frac{7}{4}} \pi^{\frac{1}{4}} \frac{e^{-2\sqrt{\pi n}}}{n^{\frac{1}{4}}} \sin\left(2\sqrt{\pi n} + \frac{5\pi}{8}\right) + O\left(\frac{e^{-2\sqrt{\pi n}}}{n^{\frac{3}{4}}}\right),$$

as $n \rightarrow \infty$.

The most remarkable aspect of this theorem lies in the fact that the arithmetic sum $\mathcal{S}_0\left(\frac{-1}{1 - e^{-i\delta}}\right)$ decays exponentially fast for $\delta \rightarrow 0^+$, while the Fourier series $h(\delta)$ is very rapidly convergent. Moreover, Theorem 3 implies that $L_2(\delta)$ can be evaluated to any given precision in a time which is independent of δ .

For a rational number $\frac{h}{k}$, $(h, k) = 1$, $k > 0$, define

$$c_0\left(\frac{h}{k}\right) = -\sum_{m=1}^{k-1} \frac{m}{k} \cot\left(\frac{\pi m h}{k}\right).$$

The value of $c_0\left(\frac{h}{k}\right)$ is an algebraic number, i.e. $c : \mathbb{Q} \rightarrow \overline{\mathbb{Q}}$, and, more precisely, $c_\ell\left(\frac{h}{k}\right)$ is contained in the maximal real subfield of the cyclotomic field of k -th roots of unity. Moreover, c_0 is odd and is periodic of period 1.

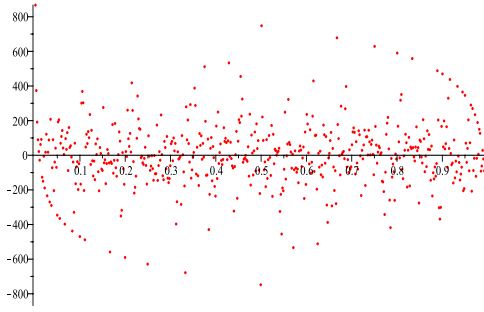


Figure 1: Graph of $c_0\left(\frac{h}{k}\right)$ for $1 \leq h < k = 541$.

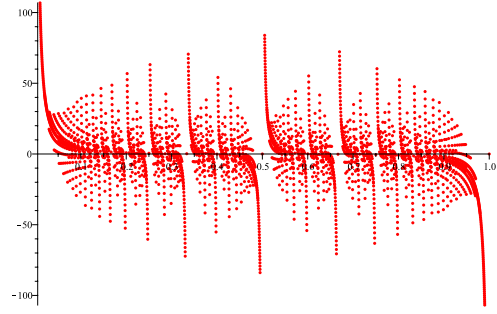


Figure 2: Graph of $c_0\left(\frac{h}{k}\right)$ for $1 \leq h \leq k \leq 100$, $(h, k) = 1$.

The cotangent sum $c_0\left(\frac{h}{k}\right)$ arises in analytic number theory in the value at $s = 0$,

$$D\left(0, \frac{h}{k}\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{h}{k}\right), \quad (12)$$

of the Estermann function, defined for $\Re(s) > 1$ by

$$D\left(s, \frac{h}{k}\right) := \sum_{n=1}^{\infty} \frac{d(n) e(nh/k)}{n^s}.$$

The Estermann function extends analytically to $\mathbb{C} \setminus \{1\}$ and satisfies a functional equation; these properties are useful in studying the asymptotics of the mean square of the Riemann zeta function multiplied by a Dirichlet polynomial (see [3]), which are needed, for example, for theorems which give a lower bound for the portion of zeros of $\zeta(s)$ on the critical line. See also [4] and [9]. The sum

$$V\left(\frac{h}{k}\right) := \sum_{m=1}^{k-1} \left\{ \frac{mh}{k} \right\} \cot\left(\frac{\pi m}{k}\right) = -c_0\left(\frac{h}{k}\right),$$

known as the Vasyunin sum, arises in the study of the Riemann zeta function by virtue of the formula:

$$\begin{aligned} \nu\left(\frac{h}{k}\right) &:= \frac{1}{2\pi\sqrt{hk}} \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left(\frac{h}{k}\right)^{it} \frac{dt}{\frac{1}{4} + t^2} \\ &= \frac{\log 2\pi - \gamma}{2} \left(\frac{1}{h} + \frac{1}{k}\right) + \frac{k-h}{2hk} \log \frac{h}{k} - \frac{\pi}{2hk} \left(V\left(\frac{h}{k}\right) + V\left(\frac{k}{h}\right) \right). \end{aligned}$$

This formula is relevant to the Nyman-Beurling-Baez-Duarte-Vasyunin ap-

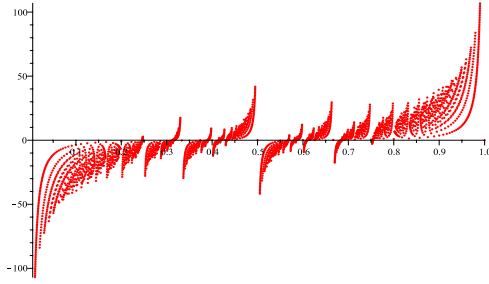


Figure 3: Graph of $V\left(\frac{h}{k}\right)$ for $1 \leq h, k \leq 100$ and $(h, k) = 1$.

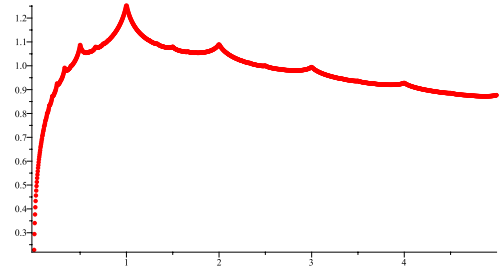


Figure 4: Graph of $\sqrt{hk} \nu\left(\frac{h}{k}\right)$ for $1 \leq h \leq 5k$, $k = 307$, $(h, k) = 1$.

proach to the Riemann hypothesis which asserts that the Riemann hypothesis is true if and only if $\lim_{N \rightarrow \infty} d_N = 0$, where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta A_N \left(\frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{4} + t^2}$$

and the inf is over all the Dirichlet polynomial $A_N(s) = \sum_{n=1}^N \frac{a_n}{n^s}$ of length N ; see [1] for a nice account of the Nyman-Beurling approach to the Riemann hypothesis with Baez-Duarte's significant contribution and see [2] and [11] for information about the Vasyunin sums, as well as interesting numerical experiments about d_N and the minimizing polynomials A_N . Thus d_N^2 is a quadratic expression in the unknown quantities a_m in terms of the Vasyunin sums.

In this paper we present a new reciprocity formula for $c_0\left(\frac{h}{k}\right)$.

Theorem 4. *Let $(h, k) = 1$, $h, k \geq 1$. Then,*

$$c_0\left(\frac{h}{k}\right) + \frac{k}{h} c_0\left(\frac{k}{h}\right) - \frac{1}{\pi h} = -\frac{i}{2} \psi_0\left(\frac{h}{k}\right). \quad (13)$$

and in particular $c_0\left(\frac{h}{k}\right) + \frac{k}{h}c_0\left(\frac{k}{h}\right) - \frac{1}{\pi h}$ is analytic on \mathbb{C}' .

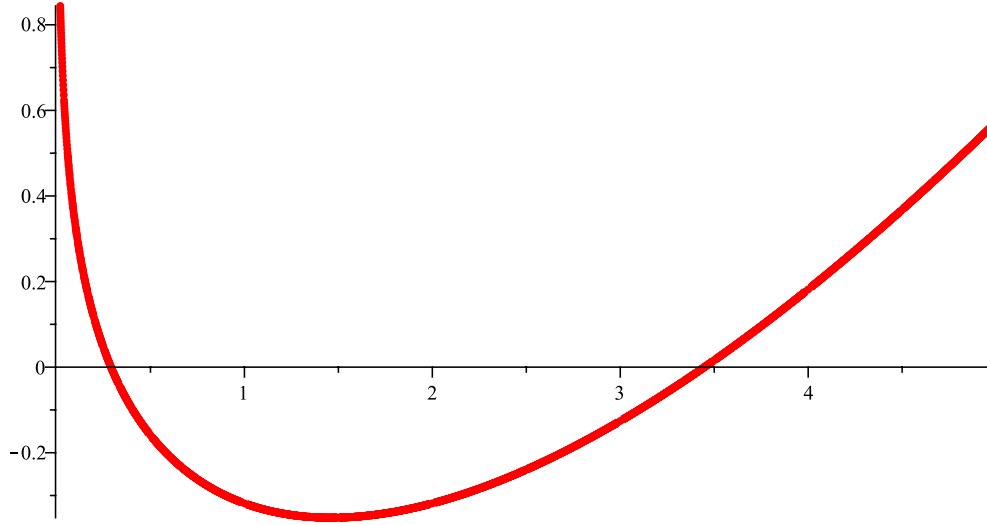


Figure 5: Graph of $c_0\left(\frac{h}{k}\right) + \frac{k}{h}c_0\left(\frac{k}{h}\right) - \frac{1}{\pi h}$ for $h \leq 5k$, $k \leq 50$ and $(h, k) = 1$.

The cotangent sum $c_0\left(\frac{h}{k}\right)$ is nowhere continuous with respect to the real topology (as can be verified intuitively looking at Figures 1 and 2), but Theorem 4 shows that $c_0\left(\frac{h}{k}\right) + \frac{k}{h}c_0\left(\frac{k}{h}\right)$ extends to an analytic function, once we subtract the correction term $\frac{1}{\pi h}$ (see figure 5). This behavior of c_0 is analogous to that of the Dedekind sum,

$$s\left(\frac{h}{k}\right) = -\frac{1}{4k} \sum_{m=1}^k \cot\left(\frac{\pi m}{k}\right) \cot\left(\frac{\pi mh}{k}\right),$$

which satisfies the well known reciprocity formula

$$s\left(\frac{h}{k}\right) + s\left(\frac{k}{h}\right) - \frac{1}{12hk} = \frac{1}{12} \left(\frac{h}{k} + \frac{k}{h} - 3\right). \quad (14)$$

Both these functions, as well as some others given below, are (nearly) examples of what Zagier calls a “quantum modular form” (see [14]).

Corollary 1. *The numbers $c_0\left(\frac{h}{k}\right)$ can be computed to within a prescribed accuracy in a time that is polynomial in $\log k$.*

This corollary descends immediately from the reciprocity formula (13) and, thanks to Euclid algorithm, the same result then holds also for $V\left(\frac{h}{k}\right)$.

Theorem 4 can be generalized to the sums

$$c_a\left(\frac{h}{k}\right) := k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \zeta\left(-a, \frac{m}{k}\right),$$

where $\zeta(s, x)$ is the Hurwitz zeta function (note that at $a = -1$ the poles of $\zeta\left(-a, \frac{m}{k}\right)$ cancel). At the non-negative integers, $a = n \geq 0$, these cotangent sums can be expressed in terms of the Bernoulli polynomials,

$$c_n\left(\frac{h}{k}\right) = -k^n \sum_{m=1}^{k-1} \cot\left(\frac{\pi mh}{k}\right) \frac{B_{n+1}\left(\frac{m}{k}\right)}{n+1},$$

most interestingly in the case when n is even, since $c_n \equiv 0$ for positive odd n .

Notice that, for all a , $c_a\left(\frac{h}{k}\right)$ is odd and periodic in $x = \frac{h}{k}$ with period 1 and, for non-negative integers a , it takes values in the maximal real subfield of the cyclotomic field of k -th roots of unity (if a is a negative integer, the same holds for $\frac{1}{\pi^{a+1}} c_a\left(\frac{h}{k}\right)$).

Like the case $a = 0$, these cotangent sums appear in the value at $s = 0$,

$$D\left(0, a, \frac{h}{k}\right) = -\frac{1}{2}\zeta(-a) + \frac{i}{2}c_a\left(\frac{h}{k}\right), \quad (15)$$

of the function $D\left(s, a, \frac{h}{k}\right)$, defined for $\Re(s) > 1$ by

$$D\left(s, a, \frac{h}{k}\right) := \sum_{n=1}^{\infty} \frac{\sigma_a(n) e(nh/k)}{n^s}.$$

Theorem 5. *Let $h, k \geq 1$, $(h, k) = 1$. Then*

$$c_a\left(\frac{h}{k}\right) - \left(\frac{k}{h}\right)^{1+a} c_a\left(\frac{-k}{h}\right) + k^a \frac{a\zeta(1-a)}{\pi h} = -i\zeta(-a)\psi_a\left(\frac{h}{k}\right). \quad (16)$$

In particular, $c_a\left(\frac{h}{k}\right)$ give an example of an “imperfect” quantum modular form of weight $1 + a$.

Note that the reciprocity formula (14) for the Dedekind sum can be obtained as a particular case of this theorem. In fact, by the reflection formula for the digamma function Ψ , we have

$$c_{-1}\left(\frac{h}{k}\right) = -\frac{1}{k} \sum_{m=1}^k \Psi\left(\frac{m}{k}\right) \cot\left(\frac{\pi m h}{k}\right) = 2\pi s\left(\frac{h}{k}\right)$$

and, since $g_{-1}(z)$ is identically zero, the reciprocity formula reduces to (14).

New formulae can be obtained by differentiating (16); for example, writing

$$c_{-1}^*\left(\frac{h}{k}\right) := \frac{1}{k} \sum_{m=1}^{k-1} \cot\left(\frac{\pi m h}{k}\right) \gamma_1\left(\frac{m}{k}\right),$$

where $\gamma_1(x)$ is the first generalized Stieltjes defined by

$$\zeta(s, x) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(x) (s-1)^n,$$

taking the derivative at -1 of (16) multiplied by k^{-a} we get the formula

$$c_{-1}^*\left(\frac{h}{k}\right) - c_{-1}^*\left(\frac{-k}{h}\right) + \frac{\zeta'(2) + \frac{\pi^2}{6}}{\pi k h} + \pi \log k \left(\frac{1}{6} \frac{k}{h} - \frac{1}{2}\right) = q\left(\frac{h}{k}\right),$$

where

$$q(z) := -\frac{1}{\pi z} \zeta'(2) + \frac{\pi}{2} (\log z + \gamma) + g'_{-1}(z)$$

is holomorphic in \mathbb{C}' .

3 The period function

In this section we give a proof of Theorem 1 and 2.

Proof of Theorem 1. Firstly, observe that the three term relation (4) follows easily from the periodicity in z of $E(a, z)$.

$\mathcal{S}_a(z)$ can be written as

$$\begin{aligned}\mathcal{S}_a(z) &= \sum_{n=1}^{\infty} \sigma_a(n) \frac{1}{2\pi i} \int_{(2+\max(0, \Re(a)))} \Gamma(s) (-2\pi i n z)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(2+\max(0, \Re(a)))} \zeta(s) \zeta(s-a) \Gamma(s) e^{\frac{\pi i s}{2}} (2\pi z)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} \zeta(s) \zeta(s-a) \Gamma(s) e^{\frac{\pi i s}{2}} (2\pi z)^{-s} ds + r_{a,M}(z),\end{aligned}\tag{17}$$

where M is any integer greater or equal to $-\frac{1}{2} \min(0, \Re(a))$ and

$$\begin{aligned}r_{a,M}(z) &:= -\frac{1}{2} \zeta(-a) + i \frac{\zeta(1-a)}{2\pi z} + i \frac{\zeta(1+a) \Gamma(1+a) e^{\frac{\pi i a}{2}}}{(2\pi z)^{1+a}} + \\ &\quad - \sum_{1 \leq n \leq M} i(-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a) (2\pi z)^{2n-1}\end{aligned}$$

is the sum of the residues encountered moving the integral (and has to be interpreted in the limit sense if some of the terms have a pole). Now, consider

$$\begin{aligned}\frac{1}{z^{1+a}} \mathcal{S}_a\left(-\frac{1}{z}\right) &= \frac{1}{z^{1+a}} \frac{1}{2\pi i} \int_{(2+\max(0, \Re(a)))} \zeta(s) \zeta(s-a) \Gamma(s) e^{\frac{\pi i s}{2}} \left(2\pi \frac{-1}{z}\right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{(2+\max(0, \Re(a)))} \zeta(s) \zeta(s-a) \Gamma(s) e^{-\frac{\pi i s}{2}} (2\pi)^{-s} z^{s-1-a} ds,\end{aligned}$$

since in this context $0 < \arg z < \pi$ and $0 < \arg \frac{-1}{z} < \pi$, so the identity $\arg \frac{-1}{z} = \pi - \arg z$ holds. Applying the functional equation to both $\zeta(s)$ and $\zeta(s-a)$ we get, after the change of variable $s \rightarrow 1-s+a$,

$$\begin{aligned}\frac{1}{z^{1+a}} \mathcal{S}_a\left(-\frac{1}{z}\right) &= -\frac{1}{2\pi} \int_{(-1+\min(0, \Re(a)))} \zeta(s-a) \zeta(s) \Gamma(s) \frac{e^{\frac{\pi i(s-a)}{2}} \cos \frac{\pi s}{2}}{\sin \frac{\pi(s-a)}{2}} (2\pi z)^{-s} ds \\ &= -\frac{1}{2\pi} \int_{(-\frac{1}{2}-M)} \zeta(s-a) \zeta(s) \Gamma(s) \frac{e^{\frac{\pi i(s-a)}{2}} \cos \frac{\pi s}{2}}{\sin \frac{\pi(s-a)}{2}} (2\pi z)^{-s} ds,\end{aligned}\tag{18}$$

since the integrand doesn't have any pole on the left of $-1 + \min(0, \Re(a))$. The theorem then follows summing (17) and (18) and using the identity

$$e^{\frac{\pi i s}{2}} + i \frac{e^{\frac{\pi i(s-a)}{2}} \cos \frac{\pi s}{2}}{\sin \frac{\pi(s-a)}{2}} = i \frac{\cos \frac{\pi a}{2}}{\sin \frac{\pi(s-a)}{2}}.$$

□

We remark that for $a = 2k + 1$, $k \geq 1$, Theorem 1 reduces to

$$E_{2k}(z) - \frac{1}{z^{2k}} E_{2k} \left(-\frac{1}{z} \right) = 0,$$

while, for $a = 1$, the theorem reduces to the well known identity

$$E_2(z) - E_2 \left(\frac{-1}{z} \right) = -\frac{12}{2\pi iz}.$$

To prove Theorem 2 we need the following lemma.

Lemma 1. *For fixed complex numbers A and α we have, as $n \rightarrow \infty$*

$$J_n := \int_0^\infty u^{n+\alpha} e^{-A\sqrt{u}} e^{-u} \frac{du}{u} = \sqrt{2\pi} e^{\frac{A^2}{8}} e^{-A\sqrt{n}} e^{-n} n^{n+\alpha-\frac{1}{2}} \left(1 - \frac{C}{\sqrt{n}} + O\left(\frac{1}{n}\right) \right),$$

where

$$C = \frac{4\alpha - 1}{8} A + \frac{A^3}{96}.$$

Proof. After the change of variable $u = nx^2$ we have

$$\begin{aligned} J_n &= 2n^{n+\alpha} \int_0^\infty x^{2\alpha-1} e^{-A\sqrt{nx}-n(x^2-2\log x)} dx \\ &= 2n^{n+\alpha} e^{-A\sqrt{n}} \int_{-1}^\infty (x+1)^{2\alpha-1} e^{-A\sqrt{nx}-n((x+1)^2-2\log(x+1))} dx \\ &= 2n^{n+\alpha} e^{-A\sqrt{n}} e^{-n} \left(1 + O\left(e^{-\frac{n\delta^2}{2}}\right) \right) \times \\ &\quad \times \int_{-\delta}^\delta (x+1)^{2\alpha-1} e^{-A\sqrt{nx}-2nx^2} \left(1 + \frac{2nx^3}{3} + O(nx^4) \right) dx, \end{aligned}$$

for any small $\delta > 0$. We can then approximate the binomial and extend the integral to \mathbb{R} at a negligible cost, getting

$$\begin{aligned} J_n &= 2n^{n+\alpha} e^{-A\sqrt{n}} e^{-n} \int_{-\infty}^\infty \left(1 + (2\alpha - 1)x + \frac{2nx^3}{3} + O(x^2 + nx^4) \right) \times \\ &\quad \times e^{-A\sqrt{nx}-2nx^2} dx. \end{aligned}$$

Evaluating the integrals, the lemma follows. □

Proof of Theorem 2. The three term relation (4) implies that

$$\begin{aligned} g_a(z+1) &= \frac{1}{(z+1)^{1+a}} \cot \frac{\pi a}{2} \zeta(-a) - \frac{1}{\pi z(z+1)^a} \zeta(1-a) + \\ &\quad + \frac{1}{\pi z(z+1)} \zeta(1-a) + g_a(z) - \frac{1}{(z+1)^{1+a}} g_a\left(\frac{z}{z+1}\right). \end{aligned}$$

Now, from the definition (5) of $g_a(z)$, it follows that

$$g_a(z) = -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a) (2\pi z)^{2n-1} + O(z^{2M+1}),$$

for any $M \geq 1$. Thus

$$\begin{aligned} g_a(z) - \frac{g_a\left(\frac{z}{z+1}\right)}{(z+1)^{1+a}} &= \\ &= -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a) (2\pi z)^{2n-1} \left(1 - \frac{1}{(z+1)^{2n+a}}\right) + O(z^{2M+1}) \\ &= 2 \sum_{m=1}^{2M} \left(\sum_{\substack{2n-1+k=m, \\ n,k \geq 1}} (-1)^{n+m} B_{2n} \zeta(1-2n-a) \frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!} (2\pi)^{2n-1} \right) z^m \\ &\quad + O(z^{2M+1}). \end{aligned}$$

Therefore,

$$g_a(z+1) = \sum_{m=0}^{2M} b_m z^m + O(z^{2M+1}),$$

where

$$\begin{aligned} b_m &:= -2 \sum_{\substack{2n-1+k=m, \\ n,k \geq 1}} (-1)^{n+k} B_{2n} \zeta(1-2n-a) \frac{\Gamma(2n+a+k)}{\Gamma(2n+a)k!(2n)!} (2\pi)^{2n-1} + \\ &\quad + (-1)^m \cot \frac{\pi a}{2} \zeta(-a) \frac{\Gamma(1+a+m)}{\Gamma(1+a)m!} + \\ &\quad + (-1)^m \left(\frac{\Gamma(1+a+m)}{\Gamma(a)(m+1)!} - 1 \right) \frac{\zeta(1-a)}{\pi}, \end{aligned}$$

and, since $g_a(z)$ is holomorphic at 1, b_m must coincide with the m -th coefficient of the Taylor series of $g_a(z)$ at 1.

Now, let's prove the asymptotic (8). Fix any $M \geq -\frac{1}{2} \min(0, \Re(a))$ and assume $m \geq 2M + 1$ and $\Re(\tau) > 0$. By the functional equation for ζ and basic properties of $\Gamma(s)$, we have

$$\begin{aligned} \frac{(2\pi)^a \tau^m}{\cos \frac{\pi a}{2}} g_a^{(m)}(\tau) &= \\ &= \frac{(-1)^m}{\pi i} \int_{(-\frac{1}{2}-2M)} \Gamma(s) \frac{\zeta(s)\zeta(s-a)}{\sin \frac{\pi(s-a)}{2}} s(s+1) \cdots (s+m-1) (2\pi)^{-s+a} \tau^{-s} ds \\ &= \frac{(-1)^m}{\pi i} \int_{(-\frac{1}{2}-2M)} \frac{\zeta(s)\zeta(s-a)}{\sin \frac{\pi(s-a)}{2}} \Gamma(s+m) (2\pi)^{-s+a} \tau^{-s} ds \\ &= \frac{(-1)^m}{\pi^3 i} \int_{(-\frac{1}{2}-2M)} \zeta(1-s)\zeta(1-s+a) \times \\ &\quad \times \Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m) \sin \frac{\pi s}{2} \left(\frac{2\pi}{\tau}\right)^s ds. \end{aligned}$$

We can see immediately that $g_a^{(m)}(\tau) \ll_a m^{-B} |\tau|^{-m} m!$ for any fixed $B > 0$, just by moving the path of integration to the line $\Re(s) = -B$ and using trivial estimates for Γ . To get a formula which is asymptotic as $m \rightarrow \infty$ we expand $\zeta(1-s)\zeta(1-s+a)$ into a Dirichlet series and integrate term-by-term; the main term arises from the first term of the sum. We have

$$g_a^{(m)}(\tau) = 2 \frac{(-\tau)^m \cos \frac{\pi a}{2}}{\pi^2 (2\pi)^a} \sum_{\ell=1}^{\infty} \frac{\sigma_{-a}(\ell)}{\ell} I_{m,a} \left(\frac{\ell}{\tau} \right),$$

where

$$I_{m,a}(x) := \frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} \Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m) \sin \frac{\pi s}{2} (2\pi x)^s ds.$$

We re-express this integral as a convolution integral. Recall that for $|\arg x| < \pi$ we have

$$\frac{1}{2\pi i} \int_{(\frac{3}{2}+2M)} \Gamma(s)\Gamma(s+a)u^{-s} ds = 2u^{\frac{a}{2}} K_a(2\sqrt{u}),$$

where K_a denotes the K-Bessel function of order a . Also,

$$\frac{1}{2\pi i} \int_{(-\frac{1}{2}-2M)} \Gamma(s+m)u^{-s} ds = u^m e^{-u}.$$

Thus,

$$I_{m,a}(x) = I_{m,a}^+(x) + I_{m,a}^-(x),$$

where

$$I_{m,a}^\pm(x) = (2\pi x)^{1+\frac{a}{2}} e^{\frac{\pm\pi ia}{4}} \int_0^\infty u^{m+\frac{a}{2}} K_a\left(2e^{\pm\frac{\pi i}{4}} \sqrt{2\pi x u}\right) e^{-u} du.$$

Now, for $|\arg z| < \frac{3}{2}\pi$

$$K_a(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4a^2 - 1}{8z} + O_a\left(\frac{1}{z^2}\right)\right),$$

as $z \rightarrow \infty$, and

$$K_{-a}(z) = K_a(z) \sim \begin{cases} 2^{a-1} \Gamma(a) z^{-a}, & \text{if } \Re(a) \geq 0, a \neq 0, \\ -\log \frac{x}{2} - \gamma, & \text{if } a = 0, \end{cases}$$

as $z \rightarrow 0$. Therefore, by Lemma 1,

$$\begin{aligned} I_{m,a}^\pm(x) &= (2\pi x)^{1+\frac{a}{2}} \frac{\pi^{\frac{1}{4}} e^{\frac{\pm\pi i(a-\frac{1}{2})}{4}}}{2^{\frac{5}{4}} x^{\frac{1}{4}}} \int_0^\infty u^{m+\frac{a}{2}-\frac{1}{4}} e^{-u-2(1\pm i)\sqrt{\pi x u}} \times \\ &\quad \times \left(1 + \frac{4a^2 - 1}{2^{\frac{9}{2}} \pi^{\frac{1}{2}} e^{\pm\frac{\pi i}{4}} \sqrt{xu}} + O_a\left(\frac{1}{u}\right)\right) du \\ &\sim 2^{\frac{1}{4}+\frac{a}{2}} \pi^{\frac{7}{4}+\frac{a}{2}} e^{\frac{\pm\pi i(a-\frac{1}{2})}{4}} x^{\frac{3}{4}+\frac{a}{2}} e^{\pm i\pi x} e^{-2(1\pm i)\sqrt{\pi x n}} e^{-m} m^{m+\frac{1}{4}+\frac{a}{2}} \times \\ &\quad \times \left(1 + \frac{\xi^\pm}{\sqrt{m}} + O\left(\frac{1}{m}\right)\right), \end{aligned}$$

where

$$\xi^\pm = -\frac{(1 \pm i)\sqrt{\pi x}(1+a)}{2} + \frac{(1 \mp i)(\pi x)^{\frac{3}{2}}}{6} + \frac{(4a^2 - 1)(1 \mp i)}{32\pi^{\frac{1}{2}}\sqrt{x}},$$

and (8) follows. □

4 An extension of Voronoi formula

Formula (10) can be proved with the same techniques used to prove Theorems 1 and 2. In this section we give an application of this formula and we discuss a similar formula for convolutions of the exponential function. We conclude the section showing how these results can be used to prove Voronoi's formula.

Applying formula (10) to $F(s) = \frac{\Gamma(\frac{s}{2})}{2\Gamma(s)}$ we get, for $\frac{\pi}{4} < \arg(z) < \frac{3}{4}\pi$,

$$\sum_{n=1}^{\infty} d(n)e^{(2\pi n z)^2} = \frac{1}{z} \sum_{n=1}^{\infty} d(n)T(4\pi n z) + R(z) + k(z), \quad (19)$$

where, for $\frac{\pi}{4} < \arg(z) < \frac{3}{4}\pi$,

$$T(z) := \frac{1}{\sqrt{\pi}i} \int_{(2)} \frac{\Gamma(s)}{\Gamma(1 - \frac{s}{2})} (-iz)^{-s} ds = \sum_{n=0}^{\infty} \frac{(iz)^n}{n! \Gamma(1 + \frac{n}{2})}$$

and

$$R(z) := \frac{1}{4} + \frac{2 \log(-4\pi iz) - 3\gamma}{8\sqrt{\pi}iz},$$

$$k(z) := \frac{1}{4\pi^2} \int_{(-\frac{1}{2})} \Gamma\left(\frac{s}{2}\right) \Gamma(1-s) \zeta(s) \zeta(1-s) z^{-s} ds.$$

Notice that we have $T(z) \ll |z|^{-B}$ for all fixed $B > 0$; moreover, $k(z)$ is holomorphic in $|\arg(z)| < \frac{3}{4}\pi$ and, if $|\arg(\tau)| < \frac{\pi}{4}$,

$$c_{\tau}(m) := \frac{k^{(m)}(\tau)}{m!} \ll |\tau|^{-m} m^{-B}$$

for all $B > 0$. In particular, if we set $z = i\delta$ with $0 < \delta \leq 1$, taking the real part of (19) we get

$$\sum_{n=1}^{\infty} d(n)e^{-(2\pi n \delta)^2} = \frac{1}{4} + \frac{-2 \log(4\pi \delta) - 3\gamma}{4\sqrt{\pi}\delta} + \Re \sum_{m=0}^{\infty} c_m \left(\frac{\sqrt{3}}{2} + i \left(\frac{1}{2} - \delta \right) \right)^m \quad (20)$$

with

$$c_m := c_{\frac{\sqrt{3}+i}{2}}(m) \ll m^{-B}.$$

for all $B > 0$.

We now state a similar formula for convolutions of the exponential function and a function that is compactly supported on $\mathbb{R}_{>0}$.

Let $g(x)$ be a compactly supported function on $\mathbb{R}_{>0}$ and let

$$W_+(z) := \int_0^\infty f\left(\frac{1}{x}\right) e(zx) \frac{dx}{x}$$

$$W_-(z) := \int_0^\infty f(x) e(zx) dx.$$

If we denote the Mellin transform of $f(x)$ with $F(s)$, then it follows that $F(s)$ is entire and that $W_+(x)$ and $W_-(x)$ can be written as in (9). In particular, since

$$F(0) = \int_0^\infty f(x) \frac{dx}{x},$$

$$F(1) = \int_0^\infty f(x) dx,$$

$$F'(1) = \int_0^\infty f(x) \log x dx,$$

formula (10) can be written as

$$\begin{aligned} \sum_{n=1}^\infty d(n)W_+(nz) - \frac{1}{z} \sum_{n=1}^\infty d(n)W_-\left(-\frac{n}{z}\right) &= \\ &= \int_0^\infty f(x) \left(\frac{1}{4x} - \frac{1}{4z} - \frac{\gamma - \log(2\pi z/x)}{2\pi iz} \right) dx + k(z) + \\ &+ \int_0^\infty f(x) \int_{(-\frac{1}{2})} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} \left(\frac{z}{x}\right)^{-s} ds \frac{dx}{2\pi x}, \end{aligned} \quad (21)$$

for $\Im(z) > 0$.

Proof of Voronoi formula. Let $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a smooth function that decays faster than any power of x and let

$$\tilde{f}(x) := 2 \int_0^\infty f(y) \cos(2\pi xy) dy$$

be the cosine transform of $f(x)$. Then, $\tilde{f}(x)$ is smooth and, by partial integration, $\tilde{f}^{(m)}(x) \ll \frac{1}{x^{2+m}}$ for all $m \geq 0$. For $0 < \Re(s) < 2$, we can define the Mellin transform of \tilde{f} ,

$$F(s) := \int_0^\infty \tilde{f}(x) x^{s-1} dx.$$

By partial integration we see that $F(s)$ extends to a meromorphic function on $\Re(s) < 2$ with simple poles at most at the non-positive integers. Also, $F(s)$ decays rapidly on vertical strips. Moreover, by Parseval's formula, for $0 < \Re(s) < 1$ we have

$$\begin{aligned} F(s) &= \frac{2}{s} \int_0^\infty f(y) (2\pi y)^{-s} \Gamma(s+1) \cos \frac{\pi s}{2} dy \\ &= \frac{2}{s} \int_0^\infty f(y) dy - 2 \int_0^\infty f(y) (\log(2\pi y) + \gamma) dy + O(s) \\ &= \frac{F_{-1}}{s} + F_0 + O(s), \end{aligned}$$

say. For $\Im(z) \geq 0$ we can define

$$\begin{aligned} W_+(z) &:= \frac{1}{2\pi i} \int_{(\frac{3}{2})} F(s) \Gamma(s) (-2\pi i z)^{-s} = \int_0^\infty \tilde{f}\left(\frac{1}{x}\right) e(zx) \frac{dx}{x}, \\ W_-(z) &:= \frac{1}{2\pi i} \int_{(\frac{3}{2})} F(1-s) \Gamma(s) (-2\pi i z)^{-s} \\ &= \int_0^\infty \left(\tilde{f}(x) - \text{Res}_{s=0} F(s) \right) e(zx) dx, \end{aligned} \tag{22}$$

with the second representation of $W_-(z)$ defined only on $\Im(z) > 0$. Since $F(s)$ is rapidly decaying at infinity, (10) holds for $\Im(z) \geq 0$ and so we can apply that formula for $z = 1$ and take the real part. By the definition of \tilde{f} , we have

$$\begin{aligned} \Re(W_+(n)) &= 2 \int_0^\infty f(y) \int_0^\infty \cos\left(\frac{2\pi y}{x}\right) \cos(nx) \frac{dx}{x} dy \\ &= \int_0^\infty f(y) (2K_0(4\pi\sqrt{ny}) - \pi Y_0(4\pi\sqrt{ny})) dy \end{aligned}$$

and

$$\begin{aligned}\Re(W_-(-n)) &= \lim_{\substack{z \rightarrow 1, \\ \Im(z) > 0}} \Re(W_-(-nz)) \\ &= \lim_{\substack{z \rightarrow 1, \\ \Im(z) > 0}} \Re \int_0^\infty \tilde{f}(x) e(-nzx) - \lim_{\substack{z \rightarrow 1, \\ \Im(z) > 0}} \Re \frac{\text{Res}_{s=0} F(s)}{-2\pi i n z} = \frac{1}{2} f(n),\end{aligned}$$

since $\text{Res}_{s=0} F(s)$ is real. Moreover, $\frac{1}{2\pi} \int_{(-\frac{1}{2})} F(s) Q(s) z^{-s} ds$ is purely imaginary on the real line, so we just need to compute

$$\begin{aligned}\Re \left(\text{Res}_{s=0,1} F(s) \Gamma(s) \zeta(s)^2 (-2\pi i)^{-s} \right) &= \\ &= \Re \left(\frac{F(1) (\gamma - \log(-2\pi i)) + F'(1)}{-2\pi i} \right. \\ &\quad \left. + \frac{-F_{-1}(\log(-2\pi i) + \gamma - 2 \log 2\pi) + F_0}{4} \right) \\ &= -\frac{f(0)}{8} - \frac{1}{2} \int_0^\infty f(y) (\log y + 2\gamma) dy,\end{aligned}$$

since $F(1) = \frac{f(0)}{2}$ and $F'(1)$ is real. This complete the proof of the theorem. \square

5 An exact formula for the second moment of $\zeta(s)$

In this section we proof the exact formula for the second moment of the Riemann zeta function.

Proof of Theorem 3. Firstly, observe that

$$L_1(\delta) = -ie^{-\frac{i\delta}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \zeta(s) \zeta(1-s) e^{i\delta s} ds.$$

The functional equation for $\zeta(s)$,

$$\zeta(1-s) = \chi(1-s) \zeta(s),$$

where

$$\chi(1-s) = (2\pi)^{-s}\Gamma(s) \left(e^{\frac{\pi is}{2}} + e^{-\frac{\pi is}{2}} \right),$$

allows us to split $L_1(\delta)$ as

$$\begin{aligned} L_1(\delta) &= -ie^{-\frac{i\delta}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \chi(1-s)\zeta(s)^2 e^{i\delta s} ds \\ &= -ie^{-\frac{i\delta}{2}} (L^+(\delta) + L^-(\delta)), \end{aligned}$$

where

$$L^\pm(\delta) = \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} (2\pi)^{-s}\Gamma(s) e^{\pm\frac{\pi is}{2}} \zeta(s)^2 e^{i\delta s} ds.$$

By Stirling's formula $L^+(\delta)$ is analytic for $\Re(\delta) > -\pi$. Moreover, by contour integration,

$$\begin{aligned} L^-(\delta) &= \int_{(2)} (2\pi)^{-s}\Gamma(s) e^{-\frac{\pi is}{2}} \zeta(s)^2 e^{i\delta s} ds - G(\delta) \\ &= J(\delta) - G(\delta), \end{aligned}$$

say, where

$$\begin{aligned} G(\delta) &:= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}} (2\pi)^{-s}\Gamma(s) e^{-\frac{\pi is}{2}} \zeta(s)^2 e^{i\delta s} ds + \\ &\quad + 2\pi i \operatorname{Res}_{s=1} \left((2\pi)^{-s}\Gamma(s) e^{-\frac{\pi is}{2}} \zeta(s)^2 e^{i\delta s} \right) \end{aligned}$$

is analytic for $\Re(\delta) < \pi$. Now, expanding $\zeta(s)^2$ into its Dirichlet series, for $\Re(\delta) > 0$ we have

$$\begin{aligned} J(\delta) &= \sum_{n=1}^{\infty} d(n) \int_{2-i\infty}^{2+i\infty} \Gamma(s) (2\pi i n e^{-i\delta})^{-s} ds \\ &= 2\pi i \mathcal{S}_0(-e^{-i\delta}) = 2\pi i \mathcal{S}_0(1 - e^{-i\delta}). \end{aligned} \tag{23}$$

By Theorem 1, we can write this as

$$J(\delta) = \frac{\log 2\pi\delta - \gamma}{1 - e^{-i\delta}} - \pi g_0(1 - e^{-i\delta}) + \frac{2\pi i}{1 - e^{-i\delta}} \mathcal{S}_0\left(\frac{-1}{1 - e^{-i\delta}}\right) + ie^{i\delta}\omega(\delta),$$

where

$$\omega(\delta) = -\frac{\log\left(\frac{1-e^{-i\delta}}{\delta}\right) - \frac{\pi i}{2}}{2 \sin\left(\frac{\delta}{2}\right)}$$

is holomorphic in $|\Re(\delta)| < \pi$. Summing up, we have

$$\begin{aligned} L_1(\delta) = & \frac{\gamma - \log 2\pi\delta}{2 \sin \frac{\delta}{2}} + \frac{\pi i}{\sin \frac{\delta}{2}} \mathcal{S}_0\left(\frac{-1}{1 - e^{-i\delta}}\right) + i\pi e^{-\frac{i\delta}{2}} g_0(1 - e^{-i\delta}) \\ & + \omega(\delta) - ie^{-\frac{i\delta}{2}} (L^+(\delta) - G(\delta)). \end{aligned} \quad (24)$$

The theorem then follows after writing

$$h(\delta) := i\pi e^{-\frac{i\delta}{2}} g_0(1 - e^{-i\delta})$$

and applying Theorem 1 and 2. \square

6 Cotangent sums

We start by recalling the basic property of $D\left(s, a, \frac{h}{k}\right)$.

Lemma 2. *For $(h, k)=1$, $k > 0$ and $a \in \mathbb{C}$,*

$$D\left(s, a, \frac{h}{k}\right) - k^{1+a-2s} \zeta(s-a) \zeta(s)$$

is an entire function of s . Moreover, $D\left(s, a, \frac{h}{k}\right)$ satisfies a functional equation,

$$\begin{aligned} D\left(s, a, \frac{h}{k}\right) = & -\frac{2}{k} \left(\frac{k}{2\pi}\right)^{2-2s+a} \Gamma(1-s+a) \Gamma(1-s) \times \\ & \times \left(\cos\left(\frac{\pi}{2}(2s-a)\right) D\left(1-s, -a, -\frac{\bar{h}}{k}\right) + \right. \\ & \left. - \cos\frac{\pi a}{2} D\left(1-s, -a, \frac{\bar{h}}{k}\right) \right), \end{aligned} \quad (25)$$

and

$$D\left(0, a, \frac{h}{k}\right) = \frac{i}{2} c_a \left(\frac{h}{k}\right) - \frac{1}{2} \zeta(-a).$$

Proof. The analytic continuation and the functional equation for $D\left(s, a, \frac{h}{k}\right)$ can be proved easily using the analogue properties for the Hurwitz zeta function and the observation that

$$D\left(s, a, \frac{h}{k}\right) = \frac{1}{k^{2s-a}} \sum_{m,n=1}^k e\left(\frac{mnh}{k}\right) \zeta\left(s-a, \frac{m}{k}\right) \zeta\left(s, \frac{n}{k}\right).$$

Moreover, applying this equality at 0, we see that

$$\begin{aligned} D\left(0, a, \frac{h}{k}\right) &= -k^a \sum_{m,n=1}^{k-1} e\left(\frac{mnh}{k}\right) \zeta\left(-a, \frac{m}{k}\right) B_1\left(\frac{n}{k}\right) - \frac{\zeta(-a)}{2} \\ &= \frac{i}{2} c_a\left(\frac{h}{k}\right) - \frac{\zeta(-a)}{2}, \end{aligned}$$

where we used

$$\sum_{n=1}^{k-1} B_1\left(\frac{n}{k}\right) \left(e\left(\frac{mh}{k}\right)\right)^n = -\frac{1}{2} \frac{1 + e\left(\frac{mh}{k}\right)}{1 - e\left(\frac{mh}{k}\right)} = -\frac{i}{2} \cot\left(\frac{\pi mh}{k}\right),$$

that can be easily obtained from the equality

$$B_1(x) = \frac{d}{dt} \left(\frac{te^{xt}}{e^t - 1} \right) \Big|_{t=0}.$$

□

Proof of Theorem 5. Firstly, observe that we can assume $0 \neq |a| < 1$, since the result extends to all a by analytic continuation. Now, taking $z = \frac{h}{k}(1 + i\delta)$, with $\delta > 0$, we have

$$\begin{aligned} \mathcal{S}_a(z) &= \sum_{n \geq 1} \sigma_a(n) e\left(n \frac{h}{k}\right) e^{-2\pi n \frac{h}{k} \delta} \\ &= \frac{1}{2\pi i} \int_{(2)} \Gamma(s) D\left(s, a, \frac{h}{k}\right) \left(2\pi \frac{h}{k} \delta\right)^{-s} ds. \end{aligned}$$

Therefore, moving the integral to $\sigma = -\frac{1}{2}$,

$$\mathcal{S}_a(z) = \frac{k^a}{2\pi h \delta} \zeta(1-a) + \frac{1}{(2\pi h \delta)^{1+a}} \zeta(1+a) \Gamma(1+a) + D\left(0, a, \frac{h}{k}\right) + O\left(\delta^{\frac{1}{2}}\right).$$

Similarly,

$$\begin{aligned}
\frac{1}{z^{1+a}} \mathcal{S}_a \left(\frac{-1}{z} \right) &= \frac{1}{z^{1+a}} \sum_{n \geq 1} \sigma_a(n) e \left(-n \frac{k}{h} \right) e^{-2\pi \frac{k}{h} \frac{\delta}{1+i\delta}} \\
&= \frac{k^a}{2\pi \delta h} \zeta(1-a) + \frac{1}{(2\pi \delta h)^{1+a}} \zeta(1+a) \Gamma(1+a) \\
&\quad - ia \frac{k^a}{2\pi h} \zeta(1-a) + \left(\frac{k}{h(1+i\delta)} \right)^{1+a} D \left(0, a, -\frac{k}{h} \right) + O \left(\delta^{\frac{1}{2}} \right).
\end{aligned}$$

In particular, as δ goes to 0, we have

$$\begin{aligned}
\mathcal{S}_a(z) - \frac{1}{z^{1+a}} \mathcal{S}_a \left(\frac{-1}{z} \right) &\longrightarrow D \left(0, a, \frac{h}{k} \right) - \left(\frac{k}{h} \right)^{1+a} D \left(0, a, -\frac{k}{h} \right) + \\
&\quad + ia \frac{k^a}{2\pi h} \zeta(1-a).
\end{aligned}$$

Applying Theorem 1, it follows that

$$\begin{aligned}
D \left(0, a, \frac{h}{k} \right) - \left(\frac{k}{h} \right)^{1+a} D \left(0, a, -\frac{k}{h} \right) &+ ia \frac{k^a}{2\pi h} \zeta(1-a) = \\
&= \frac{\zeta(-a)}{2} \left(\left(\frac{k}{h} \right)^{1+a} - 1 + \psi_a \left(\frac{h}{k} \right) \right),
\end{aligned}$$

which is equivalent to (16). \square

We conclude the paper by giving a new proof of Vasyunin formula (with a shift).

Theorem 6. *Let $(h, k) = 1$, $h, k \geq 1$. Let $|\Re(a)| < 1$. Then*

$$\begin{aligned}
\frac{1+a}{2\pi} \int_{-\infty}^{\infty} \zeta \left(\frac{1}{2} + \frac{a}{2} + it \right) \zeta \left(\frac{1}{2} + \frac{a}{2} - it \right) \left(\frac{h}{k} \right)^{-it} \frac{dt}{\left(\frac{1}{2} + \frac{a}{2} + it \right) \left(\frac{1}{2} + \frac{a}{2} - it \right)} &= \\
= -\frac{\zeta(1+a)}{2} \left(\left(\frac{k}{h} \right)^{\frac{1}{2} + \frac{a}{2}} + \left(\frac{h}{k} \right)^{\frac{1}{2} + \frac{a}{2}} \right) &+ \frac{\zeta(a)}{a} \left(\left(\frac{k}{h} \right)^{\frac{1}{2} - \frac{a}{2}} + \left(\frac{h}{k} \right)^{\frac{1}{2} - \frac{a}{2}} \right) + \\
- \left(\frac{1}{hk} \right)^{\frac{1}{2} + \frac{a}{2}} (2\pi)^a \Gamma(-a) \sin \frac{\pi a}{2} &\left(c_a \left(\frac{\bar{h}}{k} \right) + c_a \left(\frac{\bar{k}}{h} \right) \right).
\end{aligned}$$

Proof. We need to evaluate

$$\begin{aligned} & \frac{1+a}{2\pi(hk)^{\frac{1}{2}+\frac{a}{2}}} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2}+\frac{a}{2}+it\right) \zeta\left(\frac{1}{2}+\frac{a}{2}-it\right) \left(\frac{h}{k}\right)^{it} \frac{dt}{\left(\frac{1}{2}+\frac{a}{2}+it\right)\left(\frac{1}{2}+\frac{a}{2}-it\right)} = \\ & = \frac{1+a}{2\pi i} \int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)} \frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}} \frac{ds}{(s+a)(1-s)}. \end{aligned}$$

We rewrite this as

$$\begin{aligned} & \frac{1+a}{2\pi i} \int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)} \frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}} \frac{ds}{(s+a)(1-s)} = \\ & = \frac{1}{2\pi i} \int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)} \frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}} \frac{ds}{1-s} + \\ & + \frac{1}{2\pi i} \int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)} \frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}} \frac{ds}{s+a} \\ & = I_a\left(\frac{h}{k}\right) + I_a\left(\frac{k}{h}\right), \end{aligned}$$

where

$$I_a\left(\frac{h}{k}\right) := \frac{1}{2\pi i} \int_{\left(\frac{1}{2}-\frac{\Re(a)}{2}\right)} \frac{\zeta(s+a)\zeta(1-s)}{h^{s+a}k^{1-s}} \frac{ds}{1-s}.$$

The integral is not absolutely convergent, so some care is needed. One could introduce a convergence factor $e^{\delta s^2}$ and let $\delta \rightarrow 0^+$ at the end of the argument, or one could work with the understanding that the integrals are to be interpreted as $\lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT}$. We opt for the latter. Recall that $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(1-s) = ((2\pi is)^{-s} + (-2\pi is)^{-s}) \Gamma(s).$$

This leads to

$$\begin{aligned} \frac{1}{2\pi i} \int_{(2)} \frac{\chi(1-s)}{1-s} u^{-s} ds &= \frac{-1}{2\pi i} \int_{(2)} ((2\pi is)^{-s} + (-2\pi is)^{-s}) \Gamma(s) u^{-s} \\ &= \frac{-1}{2\pi i} \int_{(1)} ((2\pi is)^{-s} + (-2\pi is)^{-s}) \Gamma(s) u^{-s} \\ &= \frac{\sin 2\pi u}{\pi u}. \end{aligned}$$

Using Cauchy's theorem, the functional equation for $\zeta(s)$, and the Dirichlet series for $\zeta(s+a)\zeta(s)$, we have

$$\begin{aligned} I_a\left(\frac{h}{k}\right) &= -\operatorname{Res}_{s=1} \frac{\chi(1-s)\zeta(s+a)\zeta(s)}{h^{s+a}k^{1-s}(1-s)} - \operatorname{Res}_{s=1-a} \frac{\chi(1-s)\zeta(s+a)\zeta(s)}{h^{s+a}k^{1-s}(1-s)} + \\ &\quad + \frac{1}{\pi h^{1+a}} \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n) \sin 2\pi n \frac{h}{k}}{n} \\ &= -\frac{\zeta(1+a)}{2h^{1+a}} + \frac{\zeta(a)}{ahk^a} + \frac{1}{\pi h^{1+a}} \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n) \sin 2\pi n \frac{h}{k}}{n}. \end{aligned}$$

By the functional equation for D we see that

$$\begin{aligned} \frac{D\left(s, -a, \frac{h}{k}\right) - D\left(s, -a, -\frac{h}{k}\right)}{2i} &= \frac{2}{k} \left(\frac{k}{2\pi}\right)^{2-2s-a} \Gamma(1-s-a) \Gamma(1-s) \times \\ &\quad \times \left(\cos\left(\frac{\pi}{2}(2s+a)\right) + \cos\frac{\pi a}{2} \right) \left(D\left(1-s, a, \frac{\bar{h}}{k}\right) - D\left(1-s, a, -\frac{\bar{h}}{k}\right) \right), \end{aligned}$$

so that, defining

$$S\left(s, -a, \frac{h}{k}\right) := \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n) \sin 2\pi n \frac{h}{k}}{n^s},$$

we have

$$\begin{aligned} S\left(s, -a, \frac{h}{k}\right) &= \frac{2}{k} \left(\frac{k}{2\pi}\right)^{2-2s-a} \Gamma(1-s-a) \Gamma(1-s) \times \\ &\quad \times \left(\cos\left(\frac{\pi}{2}(2s+a)\right) + \cos\frac{\pi a}{2} \right) S\left(1-s, a, \frac{\bar{h}}{k}\right). \end{aligned} \tag{26}$$

In particular, $S\left(s, -a, \frac{h}{k}\right)$ is regular at $s=1$. Noting that

$$\lim_{s \rightarrow 1} \Gamma(1-s-a) \Gamma(1-s) \left(\cos\left(\frac{\pi}{2}(2s+a)\right) + \cos\frac{\pi a}{2} \right) = -\pi \Gamma(-a) \sin \frac{\pi a}{2}$$

and

$$S\left(0, a, \frac{\bar{h}}{k}\right) = \frac{1}{2} c_a \left(\frac{\bar{h}}{k}\right),$$

we obtain, by letting $s \rightarrow 1$ in (26), the identity

$$S\left(1, -a, \frac{h}{k}\right) = 2^a \left(\frac{\pi}{k}\right)^{1+a} \Gamma(-a) \sin \frac{\pi a}{2} c_a \left(\frac{\bar{h}}{k}\right),$$

whence

$$\sum_{n=1}^{\infty} \frac{\sigma_{-a}(n) \sin 2\pi n \frac{h}{k}}{\pi n h^{1+a}} = - \left(\frac{1}{hk}\right)^{1+a} (2\pi)^a \Gamma(-a) \sin \frac{\pi a}{2} c_a \left(\frac{\bar{h}}{k}\right).$$

Thus,

$$I_a \left(\frac{h}{k}\right) = -\frac{\zeta(1+a)}{2h^{1+a}} + \frac{\zeta(a)}{ahk^a} - \left(\frac{1}{hk}\right)^{1+a} (2\pi)^a \Gamma(-a) \sin \frac{\pi a}{2} c_a \left(\frac{\bar{h}}{k}\right)$$

and the theorem follows. \square

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